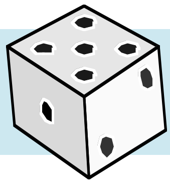
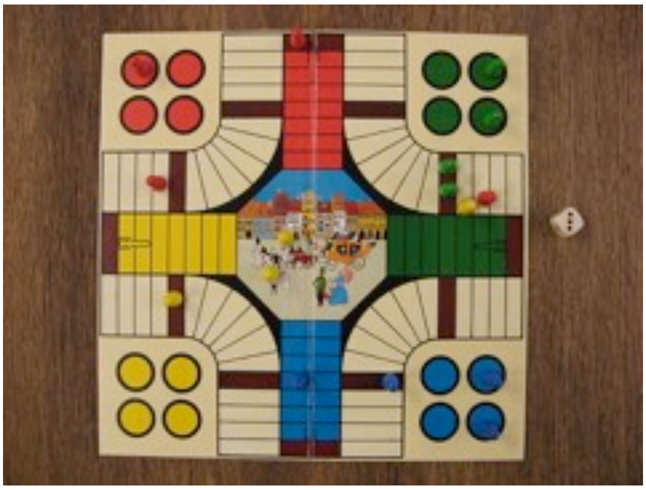
WAITING FOR A FIVE



"**Eile mit Weile**" is a board game famous especially in Switzerland. It is the simplified version of the old Indian game Pachisi which was developed in the 6th century and brought to Europe in the 19th century by the British. Each player tosses a die and then moves four pieces around the board according to the pips. If a five occurs he can move one piece out of the house and start the round with it. The winner is the player who has finished the round first with all his pieces.



Everybody has once played a board game like "Eile mit Weile" before and had the experience that it might take long to get the pieces out of the house. Sometimes one has to toss the die many times, sometimes ten or more times in a row to get a five.
It is clear that in the long run, on average every 6th throw is a five. But how is it in the short run? After five throws in a row without a five should not follow a five immediately? Or should it not at least increase the probability of tossing a five soon? Intuitively we are inclined to assume that the die has some kind of memory, or if not the die destiny has. Most fanatic gamblers believe in destiny controlling our life, especially our luck. Is that true?

1 Toss a die until a five appears and note down the number of throws needed. Repeat the procedure 50 times.

a) Calculate the relative frequency of the throws with a five at the start.
b) Now eliminate all tries with a five within the first five throws. Then calculate the relative frequency of the remaining throws showing a five at the 6th place.
c) Comment on the results of a) and b).

In our rational modern western way of thinking things have no soul and no memory. All experiments, of course invented by rational modern western brains, prove:

Even if a die does not show a five a hundred times in a row the probability of getting a five with the next toss is not rising. With each toss, unaffected by what has happened before, the probability stays at 1/6.

The die has no memory.

$\boxed{2}$ Calculate the probability to toss **no** five six times in a row.

It can happen that five or six times in a row or even more often **no** five occurs. These cases are compensated by the cases with a five at the start or at the second place and so on. In the long run, on average one toss out of six will be a five.

"On average one toss out of six" does **not** mean that after five tosses without a five that a five will always follow in the 6th toss. It means that if a die is tossed many times it can be expected that around 1/6th of all tosses will be a five.

In workshop 12 there will be the mathematical definition of the expression important in this context: **the expected value** μ .

$$\mu \text{ (number of fives with n tosses)} = n/6$$

In workshop 12 will follow:

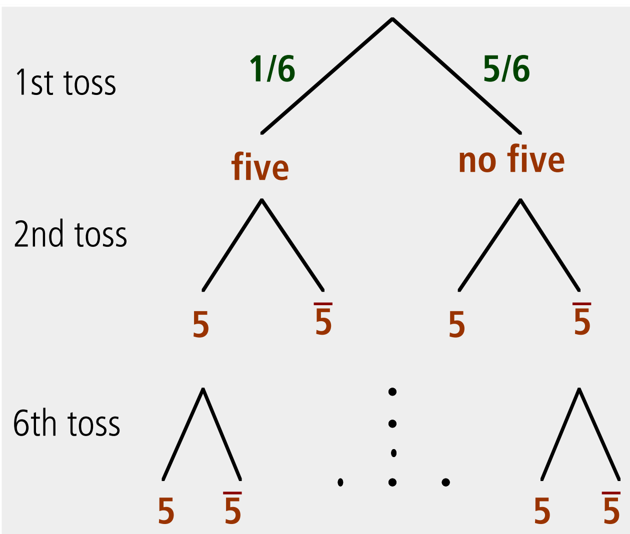
DEFINITION

Let X be a probability experiment with n numbers as outcomes: $\Omega = \{x_1, x_2, \dots, x_n\} \subset \mathbf{R}$

Then the **expected value** $\mu = \sum_{i=1}^{n} p(x_i) \cdot x_i$.

EXAMPLE

A die is tossed six times. Intuition says that in the long run, i.e. if this experiment is executed very often, we can expect an average of **one five** every six throws.



With the definition above the expected value can be calculated.

X = number of fives with 6 throws.

$$\Rightarrow \quad \Omega = \{0, 1, 2, \dots, 6\}$$

The tree diagram reflects the opinion, that the die has no memory: On each level the branching is identical, it does not matter what happened before.

$$p(0) = \text{probability that 0 fives appear} = \mathbf{1}\cdot\left(\frac{5}{6}\right)^6 = \frac{15625}{46656}$$

$$p(1) = \text{probability that 1 five appears} = \mathbf{6}\cdot\left(\frac{1}{6}\right)\left(\frac{5}{6}\right)^5 = \frac{18750}{46656}$$

$$p(2) = \text{probability that 2 fives appear} = \mathbf{15}\cdot\left(\frac{1}{6}\right)^2\left(\frac{5}{6}\right)^4 = \frac{9375}{46656}$$

$$p(3) = \text{probability that 3 fives appear} = \mathbf{20}\cdot\left(\frac{1}{6}\right)^3\left(\frac{5}{6}\right)^3 = \frac{2500}{46656}$$

$$p(4) = \text{probability that 4 fives appear} = \mathbf{15}\cdot\left(\frac{1}{6}\right)^4\left(\frac{5}{6}\right)^2 = \frac{375}{46656}$$

$$p(5) = \text{probability that 5 fives appear} = \mathbf{6}\cdot\left(\frac{1}{6}\right)^5\left(\frac{5}{6}\right)\frac{30}{46656}$$

$$p(6) = \text{probability that 6 fives appear} = \mathbf{1}\cdot\left(\frac{1}{6}\right)^6 = \frac{1}{46656}$$

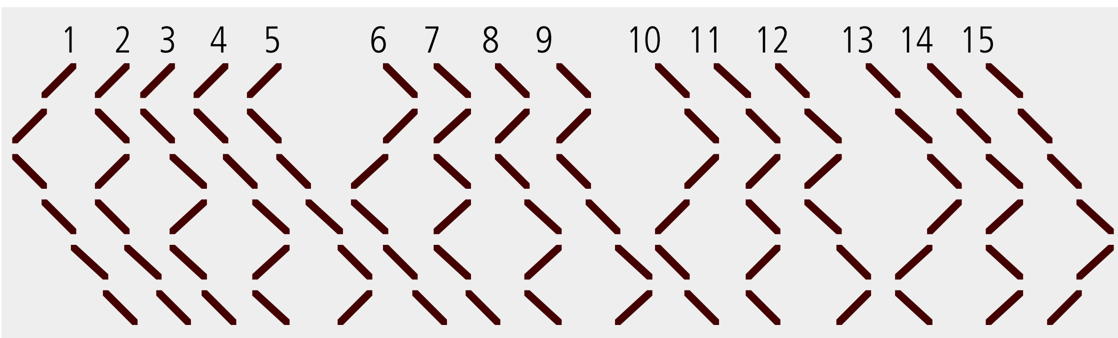
Thus

$$\mu(X) = \frac{15625}{46656}\cdot 0 + \frac{18750}{46656}\cdot 1 + \frac{9375}{46656}\cdot 2 + \frac{2500}{46656}\cdot 3 + \frac{375}{46656}\cdot 4 + \frac{30}{46656}\cdot 5 + \frac{1}{46656}\cdot 6 = 1$$

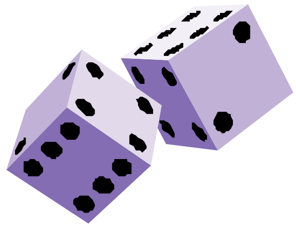
The black figures are the product of all green figures along the corresponding branch.

The blue figures are the number of branches in the tree with the corresponding number of fives. They are sometimes difficult to calculate (see in the next handout PROBABILITY DISTRIBUTIONS). At the moment the only way to evaluate them is to count all possible branches.

For example **15**: How many branches are there with two fives and four others?



3 A friend of yours claims that it is nearly impossible that out of six tossed dice there is no five. He offers you the following bet:
You toss six dice. He pays you CHF 10.– if there is no five. You on the other hand pay him CHF 1.– if there is at least one five.



Is the bet fair? If not, what should your stake be to make it fair?

A game is **fair** if the expected value for all participants is **0**.

4 Toss a coin until a "head" appears and note down the number of throws needed. Repeat the procedure 30 times.

a) Calculate the relative frequency of the throws with a "head" at the start.
b) Now eliminate all tries with a "head" at the start. Then calculate the relative frequency of the remaining throws showing a "head" at the second place.
c) Comment on the results of a) and b).

5 A coin is tossed five times.
If this experiment is repeated many times how many "heads" can be expected?

Confirm your intuitive result with a calculation.